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Bounds for turbulent shear flow

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Bounds on the transport of momentum in turbulent shear flow are derived by variational methods. In particular, variational problems for the turbulent regimes of plane Couette flow, channel flow, and pipe flow are considered. The Euler equations resemble the basic Navier–Stokes equations of motion in many respects and may serve as model equations for turbulence. Moreover, the comparison of the upper bound with the experimental values of turbulent momentum transport shows a rather close similarity. The same fact holds with respect to other properties when the observed turbulent flow is compared with the structure of the extremalizing solution of the variational problem. It is suggested that the instability of the sublayer adjacent to the walls is responsible for the tendency of the physically realized turbulent flow to approach the properties of the extremalizing vector field.

1. Introduction

A state of motion of a fluid system, which is too complex to allow a description of the velocity field in detail, is usually called turbulent. The appropriate description of a turbulent system under stationary conditions is given by the time averages of the quantities that can be measured experimentally. The goal of a theory of turbulence is the deduction of expressions for the time averages from the basic Navier–Stokes equations of motion. Equations for the time averages can be obtained by taking the corresponding averages of the basic equations. Because of the non-linearity of the basic equations, however, it is not possible to arrive at a complete system of equations for a finite set of time averages. No analytical technique seems to be available, which is successful in dealing with the coupling of the infinite set of equations for the time averages without depending on hypothetical assumptions. In addition, the solution of a large but finite number of these equations has in common with the direct computation of the turbulent field and subsequent averaging that it involves more information than is desired as a final result. For this reason a different approach is used in the present paper, which intends to give bounds for averaged quantities rather than to determine them exactly. Thus the unavoidable lack of information about the averaged quantities is expressed in the partial indeterminacy of the theoretical result.

This approach was used by Howard (1963), when he derived upper bounds for the heat transport by turbulent convection in a layer heated from below. The idea of the bounding method is to enlarge the unknown class of turbulent

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solutions of the basic equations by considering, for example, all possible fields which satisfy the boundary conditions and the energy balance for the turbulent velocity field. The maximum of the transport among the enlarged class of fields can be determined by variational techniques and provides an upper bound for the actual transport by the physically realized turbulent flow. This bound can be improved, of course, by imposing additional constraints on the class of admissible fields. A most obvious constraint is the equation of continuity. Using this constraint, we shall extend the analysis of a previous paper (Busse 1969*a*), in which bounds on the momentum transport in turbulent shear flow based solely on the energy balance have been derived. We shall refer to this paper as (I). The most interesting consequence of the constraint will be the enriched structure of the extremalizing vector fields.

We shall focus the attention on the Couette case as the simplest example for momentum transport in turbulent shear flow. After a short recapitulation of the variational problem in §2 a class of solutions of the corresponding Euler equations will be derived in §3. The solutions are characterized by a multiple boundary-layer structure, which has been discussed in detail in the paper (Busse 1969*b*) on turbulent convection, that we shall refer to as (C). The extremalizing solutions of the Euler equations have physical significance beyond their purpose of providing the upper bound on the transport of momentum. They describe a vector field which can be regarded as a model for the physically realized turbulence, since the Euler equations resemble the equations of motion in many respects. This model may be used to test various assumptions which have been introduced in heuristic theories of turbulence. There seems to exist, however, an even closer relation between the extremalizing solutions and the observed structure of turbulence, as will be shown in §4. This correspondence has been demonstrated previously, for the case of turbulent thermal convection, in Howard's paper and in (C). The experimental observations of turbulent Couette flow between parallel plates have been less extensive. Still, a close similarity can be found in all respects, for which a comparison between the extremalizing vector field and the observed turbulent flow is possible. §5 and §6 describe the application of the analogous variational problems to turbulent channel and pipe flow. The bound on the turbulent transport of momentum leads in this case to upper bounds for the mean pressure gradient and the friction coefficient at a given value of Reynolds number. The detailed observations of turbulent pipe flow provide a basis for a direct comparison between the fluctuating part of the turbulent velocity field and the extremalizing field. Although the latter is incompletely represented by the boundary-layer approximation, a correspondence is demonstrated. The striking property of the realized turbulence to approach the structure with the quality of optimal transport is discussed in §7. An explanation of this phenomenon is attempted in terms of the stability of the laminar sublayer adjacent to the wall.

2. The variational problem

We consider a homogeneous incompressible fluid between two parallel rigid plates. The plates are infinitely extended and are moving relative to each other with the velocity V_0 in the direction denoted by the constant unit vector $\mathbf{i}$. For the dimensionless description of the problem, we introduce the distance d of the plates as length scale, and d^2/ν as scale of the time. ν is the kinematic viscosity of the fluid. The Navier–Stokes equation for the velocity vector $\mathbf{V}$ is

$$\nabla^2\mathbf{V}-\nabla p = \mathbf{V}\cdot\nabla\mathbf{V}+\frac{\partial}{\partial t}\mathbf{V}. \tag{1}$$

It is convenient to introduce a Cartesian system of co-ordinates with the origin in the centre between the plates and the z-axis normal to the plates. The boundary condition for the vector $\mathbf{V}$ is given by

$$\mathbf{V} = \mp\tfrac{1}{2}Re\,\mathbf{i}, \quad \text{at} \quad z = \pm\tfrac{1}{2},$$

with $Re = V_0 d/\nu$. We assume that the velocity and the pressure are bounded everywhere, and that the average of the velocity components and their products over planes $z = \text{const.}$ exists. This average will be indicated by a bar. Accordingly, the vector $\mathbf{V}$ can be separated into two parts:

$$\mathbf{V} = \mathbf{U}+\hat{\mathbf{v}}, \quad \text{with} \quad \mathbf{U} \equiv \overline{\mathbf{V}}.$$

We wish to consider turbulent velocity fields under stationary conditions long after any change in the motion of the plates has occurred. It is reasonable to define this case by the assumption that the averages over planes $z = \text{const.}$ do not depend on time. This assumption allows the deduction (see (I)) of the following relation for the fluctuating part $\hat{\mathbf{v}}$ of the velocity field,

$$\langle|\nabla\times\hat{\mathbf{v}}|^2\rangle+\langle|\overline{\hat{\mathbf{u}}\hat{w}}-\langle\hat{\mathbf{u}}\hat{w}\rangle|^2\rangle-Re\,\mathbf{i}\cdot\langle\hat{\mathbf{u}}\hat{w}\rangle = 0. \tag{2}$$

$\hat{w}$ is the z-component of $\hat{\mathbf{v}}$, $\hat{\mathbf{u}}$ is the component parallel to the plates. The angular brackets indicate the average over the entire fluid layer. The quantity $\langle\mathbf{i}\cdot\hat{\mathbf{u}}\hat{w}\rangle$ can be regarded as the convective part of the momentum transport between the plates which is added in the case of turbulent flow to the momentum transport by viscous diffusion corresponding to the laminar solution with $\hat{\mathbf{v}} \equiv 0$.

The idea of obtaining an upper bound for the convective momentum transport is to find the maximum of $\langle\mathbf{i}\cdot\mathbf{u}w\rangle$ among a class of vector fields $\mathbf{v}$, which is defined by simple constraints, and which contains all possible solutions $\hat{\mathbf{v}}$ of the equations of motion. In the following, we shall consider the class of vector fields $\mathbf{v}$ satisfying the equation of continuity,

$$\nabla\cdot\mathbf{v} = 0, \tag{3}$$

the boundary condition, $\quad \mathbf{v} = 0, \quad \text{at} \quad z = \pm\tfrac{1}{2}, \qquad (4)$

and the relation $\quad \langle|\nabla\times\mathbf{v}|^2\rangle+\langle|\overline{\mathbf{u}w}-\langle\mathbf{u}w\rangle|^2\rangle-B\langle\mathbf{i}\cdot\mathbf{u}w\rangle = 0 \qquad (5)$

with B as given parameter. It has been shown in (I) that $\mu(B)$ is a monotonic function and that the variational problem can be formulated in the following form: Given $\mu \geqslant 0$, find the minimum $B(\mu)$ of the functional,

$$\mathscr{B}(\mathbf{v},\mu) \equiv \frac{\langle|\nabla\times\mathbf{v}|^2\rangle}{\langle\mathbf{u}\cdot\mathbf{i}w\rangle} + \mu\frac{\langle|\overline{\mathbf{u}w} - \langle\mathbf{u}w\rangle|^2\rangle}{\langle\mathbf{u}\cdot\mathbf{i}w\rangle^2}, \tag{6}$$

among all vector fields $\mathbf{v}$ that satisfy the conditions (3) and (4).

Since the functional (6) is homogeneous of degree zero, the amplitude of the solution $\mathbf{v}$ remains undetermined. This fact is used to satisfy relation (5) by imposing

$$\langle\mathbf{u}\cdot\mathbf{i}w\rangle = \mu \tag{7}$$

as normalization condition. Since the variational problem does not depend on this condition, more convenient conditions will be used in the following in place of (7). In order to eliminate the constraint (3), we introduce a representation of $\mathbf{v}$ in terms of two scalar functions, which holds for arbitrary divergence-free vector fields,

$$\mathbf{v} = \nabla\times(\nabla\times\mathbf{k}v) + \nabla\times\mathbf{k}\psi. \tag{8}$$

$\mathbf{k}$ is the unit vector in the direction of the z-axis. The Euler equations for the variables v and ψ corresponding to an extremum of the variational functional (6) will be considered in § 3.

3. The extremalizing solutions

For the solution of the variational problem, we introduce the hypothesis that the minimizing functions v, ψ are independent of x. The question how far this hypothesis can be justified will be discussed at the end of this section. For x-independent fields $\mathbf{v}$, the functional (6) becomes similar to the corresponding functional in the case of thermal convection. The similarity becomes apparent if

$$\theta \equiv \frac{\partial}{\partial y}\psi = \mathbf{u}\cdot\mathbf{i} \tag{9}$$

is introduced as new variable in place of ψ, and if the positive term

$$\left\langle\left(\frac{\partial^2 v}{\partial y\,\partial z}\frac{\partial^2 v}{\partial y^2} - \left\langle\frac{\partial^2 v}{\partial y\,\partial z}\frac{\partial^2 v}{\partial y^2}\right\rangle\right)^2\right\rangle \tag{10}$$

is neglected in the denominator of the second term on the right-hand side of (6). It will be found that the term (10) vanishes automatically for the solutions of the Euler equations. Using the above assumptions, we obtain as functional in place of (6)

$$\mathscr{B}(v,\theta;\mu) \equiv \frac{\langle|\mathbf{k}\times\nabla\nabla^2 v|^2\rangle + \langle|\nabla\theta|^2\rangle}{\langle w\theta\rangle} + \mu\frac{\langle(\overline{w\theta} - \langle w\theta\rangle)^2\rangle}{\langle w\theta\rangle^2}, \tag{11}$$

which is closely related to functional in the case of convection with θ corresponding to the fluctuating part of the temperature field. Here and in the following w is used as abbreviation for $(\partial^2/\partial y^2)v$. The dissipation term, that is the denominator of the first term on the right-hand side of (11), is the only term that depends on the ratio D of the amplitudes of v and θ when the product of both amplitudes is kept fixed. It reaches a minimum when D is chosen in such a way that

$$\langle|\mathbf{k}\times\nabla\nabla^2 v|^2\rangle+\langle|\nabla\theta|^2\rangle = 2\langle|\mathbf{k}\times\nabla\nabla^2 v|^2\rangle^{\frac{1}{2}}\langle|\nabla\theta|^2\rangle^{\frac{1}{2}} \tag{12}$$

holds. Hence the minimum $B(\mu)$ of the functional (11) is identical with the minimum of the modified functional, in which the dissipation term is replaced according to relation (12). In order to apply the mathematical results to more general cases of shear flow turbulence, we introduce in addition the positive function $h(z)$ with the properties

$$\langle(h(z))^2\rangle = 1, \quad h(\tfrac{1}{2}) = h(-\tfrac{1}{2}) = h_0.$$

Thus we obtain, as final version of the variational functional,

$$\mathscr{B}(v,\theta;\mu) \equiv 2\,\frac{\langle|\mathbf{k}\times\nabla\nabla^2 v|^2\rangle^{\frac{1}{2}}\langle|\nabla\theta|^2\rangle^{\frac{1}{2}}}{\langle w\theta h\rangle}+\mu\,\frac{\langle(\overline{w\theta}-h\langle w\theta h\rangle)^2\rangle}{\langle w\theta h\rangle^2}. \tag{13}$$

The Couette case considered in §3 and §4 is recovered by specifying

$$h(z) \equiv h_0 = 1$$

in the definition (13). Since the functional (13) is homogeneous of degree zero with respect to v, as well as with respect to θ, it is convenient to impose as normalization conditions

$$\langle w\theta h\rangle = 1, \quad \langle w^2\rangle = \langle\theta^2\rangle. \tag{14}$$

The form of the functional (13) suggests that the Euler equations have solutions for which the y- and the z-dependence separate. It can be shown that a solution of this kind determines the minimum $B(\mu)$ for sufficiently small values of μ. The Euler equations for $\mu = 0$ are

$$\left.\begin{aligned} D^{-1}\nabla^4 w-\tfrac{1}{2}hB(0)\frac{\partial^2}{\partial y^2}\theta &= 0,\\ D\nabla^2\theta+\tfrac{1}{2}hB(0)w &= 0,\end{aligned}\right\} \tag{15}$$

with D denoting the ratio $\langle|\mathbf{k}\times\nabla\nabla^2 v|^2\rangle^{\frac{1}{2}}\langle|\nabla\theta|^2\rangle^{-\frac{1}{2}}$. The eigenvalue problem described by (15), together with the boundary conditions

$$w = \frac{\partial}{\partial t}w = \theta = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2},$$

is identical with the problem describing the instability of a layer heated from below with respect to two-dimensional disturbances. The minimum eigenvalue in the Couette case is

$$B(0) = 2\surd 1708.$$

For larger values of μ, solutions of a more general form have to be considered:

$$v = v^{(N)} \equiv \sum_{n=1}^{N}\phi_n(y)\,w_n(z)/\alpha_n^2, \quad \theta = \theta^{(N)} \equiv \sum_{n=1}^{N}\phi_n(y)\,\theta_n(z), \tag{16}$$

with $\phi_n(y)$ satisfying the relations,

$$\frac{d^2}{dy^2}\phi_n(y) = -\alpha_n^2\phi_n(y), \quad \langle\phi_n^2\rangle = 1.$$

The dependence on N of the variables ϕ_n, w_n and θ_n has not been denoted explicitly. It will be indicated by an upper index N when it becomes necessary to distinguish different solutions. It is proposed that the solution of the variational problem can be found among the class of solutions (16), which includes the separable solutions corresponding to $N = 1$. There are some reasons, though no formal proof, which suggest that this hypothesis is correct. A discussion of the analogous problem in the convection case can be found in (C).

In the following we shall restrict ourselves to solutions of the form (16) in the limit of large μ, in which case boundary-layer techniques become applicable. For large μ the last term in the definition (13) becomes dominant. It reaches its minimum when $\overline{w\theta}$ becomes equal to $h(z)$. For a minimum of $\mathscr{B}$ this relation cannot be strictly fulfilled, since the dissipation term diverges owing to the boundary conditions. Hence a balance between the two terms on the right-hand side of (13), with $\overline{w\theta}$ tending to zero in a thin boundary layer, will be optimal. Since the dissipation term reaches a minimum when the derivatives with respect to the z- and the y-dependence are of the same order, a sequence of boundary layers, which allow for a transition from the interior scale to the scale of the boundary layer of $\overline{w\theta}$, can be expected. In the N boundary layers, each of which is characterized by a scale of the order μ^{-r_n}, the functions w_n, θ_n grow from their boundary values,

$$w_n = \frac{d}{dz} w_n = \theta_n = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}.$$

On the larger scale of the order $\mu^{-r_{n-1}}$ where w_{n-1}, θ_{n-1} are growing w_n, θ_n decay to zero in such a way that the relation,

$$w_n \theta_n + w_{n-1} \theta_{n-1} \approx h_0, \tag{17}$$

is approximately satisfied throughout all but the Nth boundary layer. Physically, this boundary-layer structure may be interpreted as the mechanism by which eddies of different scale relieve one another in carrying the transport of momentum. The mathematical description of the structure has to take into account that the amplitudes of w_n and θ_n may be of different orders:

$$\left.\begin{aligned} w_n(z) &= \mu^{-s_n}\tilde{w}_n(\zeta_{n-1}) \\ \theta_n(z) &= \mu^{s_n}\tilde{\theta}_n(\zeta_{n-1}) \end{aligned}\right\} \quad \text{for} \quad (\tfrac{1}{2} \mp z) \approx O(\mu^{-r_{n-1}}),$$

$$\left.\begin{aligned} w_n(z) &= \mu^{-p_n}\hat{w}_n(\zeta_n) \\ \theta_n(z) &= \mu^{p_n}\hat{\theta}_n(\zeta_n) \end{aligned}\right\} \quad \text{for} \quad (\tfrac{1}{2} \mp z) \approx O(\mu^{-r_n}).$$

The respective boundary-layer co-ordinates ζ_n are defined by

$$\zeta_n \equiv \mu^{r_n}(\tfrac{1}{2} \mp z).$$

The above description includes the interior region with $r_0 = 0$ and $\zeta_0 = z$ if the relation (17) is replaced in the case $n = 1$ by

$$\tilde{w}_1 \tilde{\theta}_1 \approx h(z) \quad \text{for} \quad (\tfrac{1}{2} \mp z) \approx O(1). \tag{18}$$

Anticipating that the contributions to the integrals $\langle w^2\rangle$ and $\langle \theta^2\rangle$ from the boundary layers do not exceed the contributions from the interior, we require $s_0 = 0$ in accordance with condition (14). We set

$$\alpha_n^2 = \mu^{q_n} b_n^2,$$

and assume that b_n^2 as well as $\tilde{w}_n, \hat{w}_n, \tilde{\theta}_n, \hat{\theta}_n$ are independent of μ. Then the boundary approximation of the functional $\mathscr{B}$ is given by

$$\begin{aligned} \hat{\mathscr{B}}(v^{(N)}, \theta^{(N)}; \mu) &= 2\mu^{1-r_N} \int_0^\infty (h_0 - \hat{w}_N \hat{\theta}_N)^2 d\zeta_N \langle \tilde{w}_1 \tilde{\theta}_1 h\rangle^{-2} \\ &+ 4\left\{\sum_{n=1}^{N} \frac{\mu^{3r_n - 2p_n - q_n}}{b_n^2} \int_0^\infty \hat{w}_n''^2 d\zeta_n + \sum_{n=1}^{N} b_n^2 \mu^{q_n - r_{n-1} - 2s_n} \int_0^\infty \tilde{w}_n^2 d\zeta_{n-1} + \mu^{q_1} b_1^2 \frac{\langle \tilde{w}_1^2\rangle}{2}\right\}^{\frac{1}{2}} \\ &\times \left\{\sum_{n=1}^{N} \mu^{r_n + 2p_n} \int_0^\infty \hat{\theta}_n'^2 d\zeta_n + \sum_{n=2}^{N} b_n^2 \mu^{q_n - r_{n-1} + 2s_n} \int_0^\infty \tilde{\theta}_n^2 d\zeta_{n-1} + \mu^{q_1} b_1^2 \frac{\langle \tilde{\theta}_1^2\rangle}{2}\right\}^{\frac{1}{2}} \langle \tilde{w}_1 \tilde{\theta}_1 h\rangle^{-1}. \end{aligned} \tag{19}$$

It is readily seen, by analogy with the analysis in (C), that the minimum of $\hat{\mathscr{B}}$ with respect to the exponents of μ is reached for

$$r_n = \frac{1-4^{-n}}{2-4^{-N}}, \quad q_n = \frac{2-4\cdot 4^{-n}}{2-4^{-N}}, \quad 2p_n = \frac{4^{-n}}{2-4^{-N}}, \quad s_n = 0.$$

Accordingly, the μ-dependence of the minimum $\hat{B}(\mu)$ of $\hat{\mathscr{B}}$ can be written in the form,

$$\hat{B}^{(N)}(\mu) = F(N) \mu^{1/(2-4^{-N})}. \tag{20}$$

The Euler equations for $\tilde{w}_1$, $\tilde{\theta}_1$ corresponding to an extremum of the functional (13) with respect to the interior dependence of w and θ are

$$\left.\begin{aligned} D^{-1} b_1^2 \tilde{w}_1 - \left\{\mu^{r_N}(h - \tilde{w}_1\tilde{\theta}_1) + \tfrac{1}{2}h\left(F(N) + \int_0^\infty (h_0 - \hat{w}_N\hat{\theta}_N)^2 d\zeta_N\right)\right\}\tilde{\theta}_1 = 0, \\ D b_1^2 \tilde{\theta}_1 - \left\{\mu^{r_N}(h - \tilde{w}_1\tilde{\theta}_1) + \tfrac{1}{2}h\left(F(N) + \int_0^\infty (h_0 - \hat{w}_N\hat{\theta}_N)^2 d\zeta_N\right)\right\}\tilde{w}_1 = 0, \end{aligned}\right\} \tag{21}$$

where D denotes the ratio between the first and the second square root in (19). Since the contributions from the boundary layers to $\langle w^2\rangle$ and $\langle\theta^2\rangle$ are negligible, the normalization condition (14) requires that

$$\langle \tilde{w}_1^2\rangle = \langle \tilde{\theta}_1^2\rangle.$$

This condition determines $D = 1$ in accordance with equations (21). Hence the parameter D can be neglected in the following, where the Euler equations for $\hat{w}_n$, $\hat{\theta}_n$, $\tilde{w}_n$ and $\tilde{\theta}_n$ are considered:

$$\left.\begin{aligned} \hat{w}_n^{\text{iv}} - \mu^{r_N - r_n}(h_0 - \hat{w}_n\hat{\theta}_n - \tilde{w}_{n+1}\tilde{\theta}_{n+1})\hat{\theta}_n = 0, \\ \hat{\theta}_n'' + \mu^{r_N - r_n}(h_0 - \hat{w}_n\hat{\theta}_n - \tilde{w}_{n+1}\tilde{\theta}_{n+1})\hat{w}_n = 0, \end{aligned}\right\} \quad (n = 1, \ldots, N), \tag{22}$$

$$\left.\begin{aligned} b_{n+1}^2 \tilde{w}_{n+1} - \mu^{r_N - r_n}(h_0 - \hat{w}_n\hat{\theta}_n - \tilde{w}_{n+1}\tilde{\theta}_{n+1})\tilde{\theta}_{n+1} = 0, \\ b_{n+1}^2 \tilde{\theta}_{n+1} - \mu^{r_N - r_n}(h_0 - \hat{w}_n\hat{\theta}_n - \tilde{w}_{n+1}\tilde{\theta}_{n+1})\tilde{w}_{n+1} = 0, \end{aligned}\right\} \quad (n = 1, \ldots, N-1). \tag{23}$$

Here and in the following discussion, $\tilde{w}_{N+1}$ and $\tilde{\theta}_{N+1}$ are assumed to be replaced by zero. In the region where $\tilde{w}_{n+1}$, $\tilde{\theta}_{n+1}$ do not vanish, (23) and (21) yield

$$h_0 - \hat{w}_n \hat{\theta}_n - \tilde{w}_{n+1}\tilde{\theta}_{n+1} = \mu^{r_n - r_N} b_{n+1}^2, \quad \text{for} \quad n = 1, \ldots, N-1, \tag{24a}$$

$$h(z) - \tilde{w}_1 \tilde{\theta}_1 = \mu^{-r_N}\left(b_1^2 - \tfrac{1}{2}h(z)\left(F(N) + \int_0^\infty (h_0 - \hat{w}_N\hat{\theta}_N)^2 d\zeta_N\right)\right), \tag{24b}$$

which indicate that the relations (17) and (18) cannot be satisfied exactly. The corresponding contributions in the second term on the right-hand side of the expression (13), however, can be neglected in the boundary-layer approximation (19). Further consequences of equations (22), (23), (24) are

$$b_n^{-2}\int_0^\infty \hat{w}_n''^2 d\zeta_n = \int_0^\infty \hat{\theta}'^2 d\zeta_n, \quad \text{for} \quad n = 1, \ldots, N, \tag{25a}$$

$$\tilde{w}_{n+1}^2 = \tilde{\theta}_{n+1}^2 = h_0 - \hat{w}_n\hat{\theta}_n, \quad \text{for} \quad n = 1, \ldots, N-1, \tag{25b}$$

$$\tilde{\theta}_1 = \tilde{w}_1 = (h(z))^{\frac{1}{2}}. \tag{25c}$$

The relations (24) can be used to simplify (22). After eliminating $\hat{\theta}_n$, we obtain

$$\hat{w}_n^{\text{vi}} - b_n^2 b_{n+1}^4 \hat{w}_n = 0, \quad \text{for} \quad n = 1, \ldots, N-1. \tag{26}$$

This equation holds in the region where $\hat{w}_n\hat{\theta}_n$ is growing up to the value h_0. As soon as this value is reached, $\tilde{w}_{n+1}, \tilde{\theta}_{n+1}$ vanish and relations (24) become incorrect. Instead, the condition $\hat{w}_n\hat{\theta}_n \approx 1$ becomes valid. The identical problem has been discussed in detail in the convection case (C). For the following we need only the results

$$\left.\begin{aligned} b_n^{-2}\int_0^\infty \hat{w}_n''^2 d\zeta_n + b_{n+1}^2\int_0^\infty \tilde{w}_{n+1}^2 d\zeta_n &= \left(\frac{b_{n+1}^4}{b_n}\right)^{\frac{1}{3}} h_0 3\beta, \quad \text{for} \quad n = 1, \ldots, N-1, \\ b_N^{-2}\int_0^\infty \hat{w}_N''^2 d\zeta_N = \frac{1}{4}\int_0^\infty (h_0 - \hat{w}_N\hat{\theta}_N)^2 d\zeta_N &= \left(\frac{h_0^2}{b_N}\right)^{\frac{1}{3}} h_0\sigma. \end{aligned}\right\} \tag{27}$$

The constants σ and β have been computed by Howard (1963) and in (C),

$$\sigma = 0{\cdot}337, \quad 3\beta = 1{\cdot}847.$$

The relations (27), (25) yield

$$\hat{\mathscr{B}}(v^{(N)}, \theta^{(N)}; \mu) = \mu^{1/(2-4^{-N})} h_0\left\{12\sigma(h_0^2/b_N)^{\frac{1}{3}} + 12\beta\sum_{n=1}^{N-1}\left(\frac{b_{n+1}^4}{b_n}\right)^{\frac{1}{3}} + 2b_1^2 h_1\right\}, \tag{28}$$

where h_1 is defined by

$$h_1 h_0 \equiv \langle |h(z)| \rangle = \langle \tilde{\theta}_1^2 \rangle = \langle \tilde{w}_1^2 \rangle.$$

The right side of (28) reaches a minimum when the derivatives with respect to the variables b_n vanish. This requirement yields N equations,

$$b_1 = \left(\frac{b_2}{b_1}\right)^{\frac{4}{3}} \beta/h_1,$$

$$\left(\frac{b_{n+1}^4}{b_n}\right)^{\frac{1}{3}} = 4\left(\frac{b_n^4}{b_{n-1}}\right)^{\frac{1}{3}}, \quad \text{for} \quad n = 2, \ldots, N-1,$$

$$\sigma\left(\frac{h_0^2}{b_N^4}\right)^{\frac{1}{3}} = 4\beta\left(\frac{b_N}{b_{N-1}}\right)^{\frac{1}{3}},$$

with the solutions

$$\left.\begin{aligned} b_1 &= \{(4^{\frac{4}{3}}\beta/h_1)^{1-4^{-N}}(\sigma/\beta)^{\frac{3}{4}}\,h_0^{\frac{1}{2}}4^{-}\}^{1/(2-4^{-N})}, \\ b_{n+1} &= (b_1 h_1/4^{\frac{4}{3}}\beta)^{1-4^{-n}} b_1 4^n, \quad \text{for} \quad n = 1, \ldots, N-1. \end{aligned}\right\} \tag{29}$$

Accordingly, the minimum $\hat{B}^{(N)}(\mu)$ is given by

$$\hat{B}^{(N)}(\mu) = \mu^{1/(2-4^{-N})}F(N) = \mu^{1/(2-4^{-N})}2h_0 h_1(2-4^{-N})4^N b_1^2. \tag{30}$$

Before we draw conclusions from this result we return to the hypothesis that the minimizing solution of the functional (6) is independent of x. In the limit $\mu \to 0$, this hypothesis has been established by a proof given by Joseph (1966). For large values of μ, the hypothesis still seems to be correct, though difficult to prove. It is of interest to consider the vector fields $\mathbf{v}$ independent of y. In this case u_y vanishes, since only a positive term would be added to the functional (6) otherwise. The equation of continuity is satisfied if

$$u_x \equiv \mathbf{i} \cdot \mathbf{u} = \frac{\partial}{\partial t}\phi, \quad w = -\frac{\partial}{\partial x}\phi$$

is assumed. Solutions analogous to solutions (16) with a multiple boundary-layer structure can be discussed. We shall restrict our attention to the case $N = 1$, with

$$\frac{\partial}{\partial x} \approx i\alpha.$$

Assuming that the boundary-layer thickness is of the order μ^{-r}, and that α is of the order μ^{-q}, the functional (6) yields terms of the order

$$\mu^{3r-q}, \quad \mu^{q+r}, \quad \mu^{3q-r}, \quad \text{and} \quad \mu^{1-r},$$

which indicate that the functional will grow at least like $\mu^{\frac{2}{3}}$. This consideration can be extended to the case $N > 1$ with the result that the exponents of μ of $\hat{B}(\mu)$ are always larger in the y-independent case than in the x-independent case. Since the general case can be regarded as a combination of both cases, it seems unlikely that the absolute minimum of the functional (6) at given value is reached by x-dependent solutions.

Another reason suggesting the validity of the hypothesis is the fact that the absolute minimum among the class (30) of relative minima is given, in limit $\mu \to \infty$, by

$$\hat{B}^{(\infty)}(\mu) = h_0^{\frac{3}{2}}4^{\frac{7}{3}}(\sigma^3\beta)^{\frac{1}{4}}\mu^{\frac{1}{2}}, \tag{31}$$

which shows the same dependence with respect to μ as the solution of the variational problem without the constraint of the continuity equation described in (I). In the Couette case, the exact solution of this problem yields

$$B_c(\mu) = (2\mu)^{\frac{1}{2}}\tfrac{8}{3}, \quad \text{for} \quad \mu \to \infty,$$

which differs from the value of (31) for $h_0 = h_1 = 1$ only by a factor of 0·38.

4. The bound on the momentum transport

The functions $\hat{B}^{(N)}(\mu)$ are asymptotic expressions for the exact minima $B^{(N)}(\mu)$ in the case when μ tends to infinity. Because of the monotonic dependence of $B^{(N)}(\mu)$ on μ, it is reasonable to assume that the boundary-layer representation $\hat{B}^{(N)}(\mu)$ approximates the exact dependence closely at large, but finite, values of

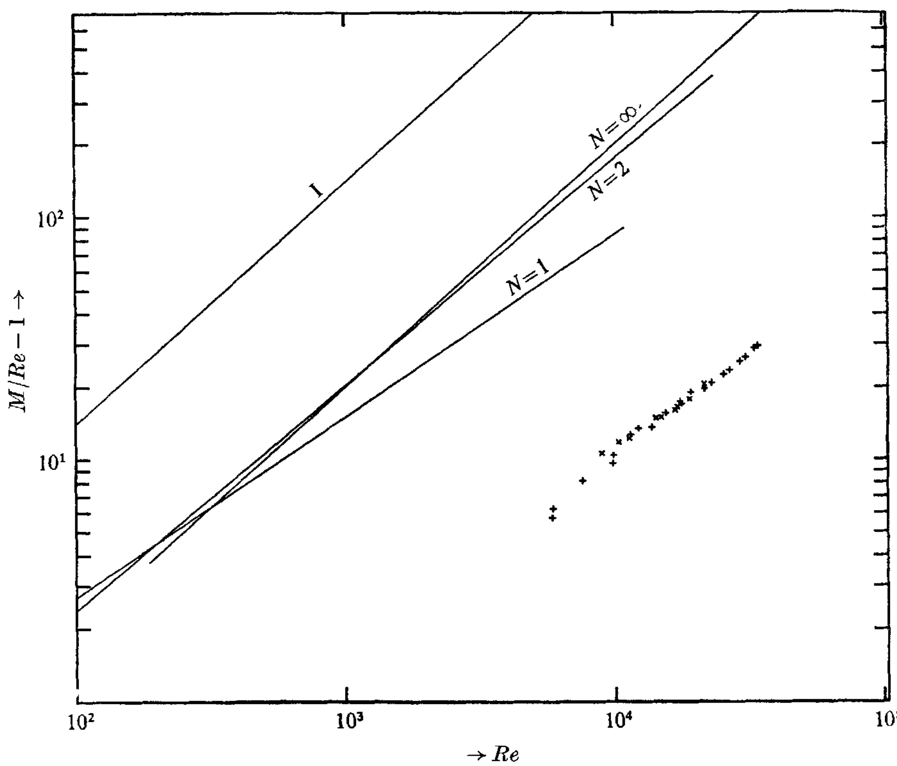

FIGURE 1. The upper bound for the momentum transport M by turbulent Couette flow. The bound for $M/Re-1$ has been plotted in comparison with the experimental values by Reichardt (1959) for water (×) and for air (+). The line labelled I describes the asymptotic bound derived without the constraint of the equation of continuity.

μ. To obtain an upper bound for the momentum transport M in turbulent Couette flow, the inverse function $\hat{\mu}^{(N)}(B)$ of $\hat{B}^{(N)}(\mu)$ has to be considered. The maximum of $\hat{\mu}^{(N)}(Re)$ provides the upper bound at a given value of Re,

$$M \equiv Re + \langle \hat{\mathbf{u}} \cdot \mathbf{i}\hat{w} \rangle \leqslant Re + \max_N(\hat{\mu}^{(N)}(Re)).$$

Figure 1 shows that the upper bound is given by one after the other of the functions $\hat{\mu}^{(N)}(Re)$, starting with $\hat{\mu}^{(1)}(Re)$ at low values of Re. The comparison with the experimental data by Reichardt (1959) shows that the turbulent momentum transport amounts only to about $\frac{1}{10}$ of the upper bound, yet it exhibits a similar dependence on the Reynolds number Re.

The similarity between the solution of the variational problem and the observed turbulent flow is even more pronounced in other aspects of the problem.

It is interesting, for instance, to calculate the profile of the mean flow corresponding to the extremalizing solution in the interior,

$$\frac{dU_x^{(N)}}{dz} = \mu(\overline{w^{(N)}\theta^{(N)}} - \langle w^{(N)}\theta^{(N)}\rangle) - B^{(N)}(\mu). \tag{32}$$

According to relation (24*b*), the right side is independent of z in the Couette case, and yields $-\frac{1}{4}\hat{B}^{(\infty)}(\mu)$ as μ tends to infinity. Hence, the profile of the extremalizing solution has a constant shear, which amounts to $\frac{1}{4}$ of the shear of the laminar

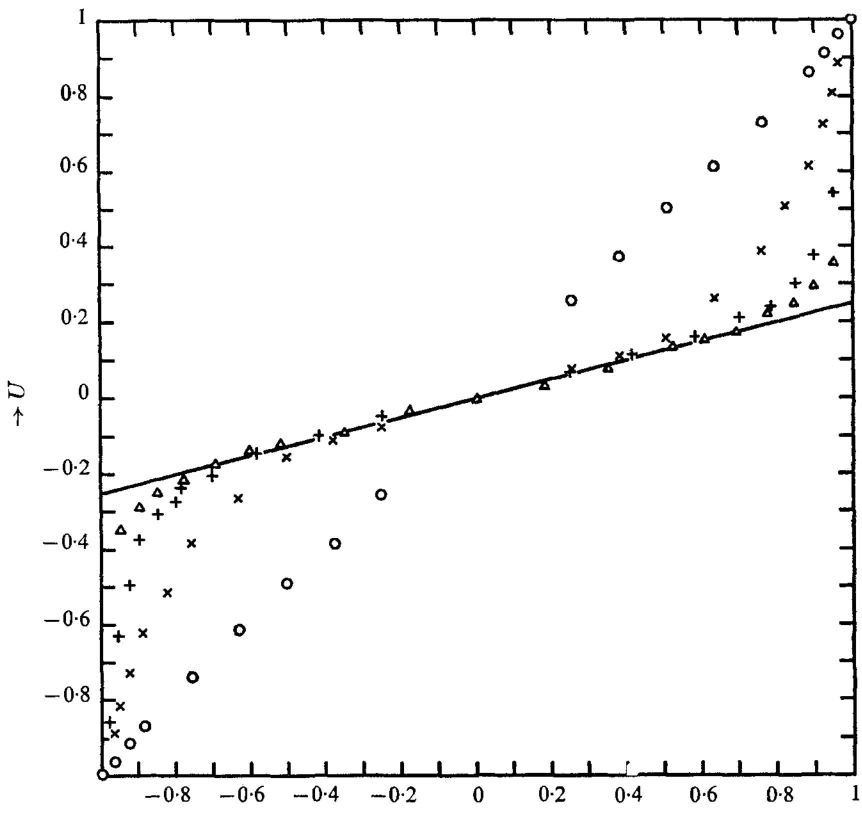

FIGURE 2. The mean velocity in plane Couette flow measured by Reichardt (1959) at $Re = 1200$ (○), $Re = 2900$ (×), $Re = 5900$ (+), and $Re = 34000$ (△). The straight line describes the asymptotic profile corresponding to the extremalizing solution of the variational problem.

solution. Since it seems physically reasonable to assume that the turbulent flow nearly wipes out the shear in the interior, it is surprising to find that the measurements of Reichardt (1959) reflect the '$\frac{1}{4}$-law' as shown in figure 2. In this respect the Couette case differs from the case of thermal convection, in which the mean temperature gradient becomes vanishingly small in the interior compared with the temperature difference applied at the boundaries.

The comparison with respect to the fluctuating velocity field is more difficult mainly because of the incomplete description by the boundary-layer method.

The fact that the minimizing vector field is y-independent corresponds to the observation (Townsend 1956) that the turbulent momentum transport is carried predominantly by wall-attached eddies with a streamwise axis. The

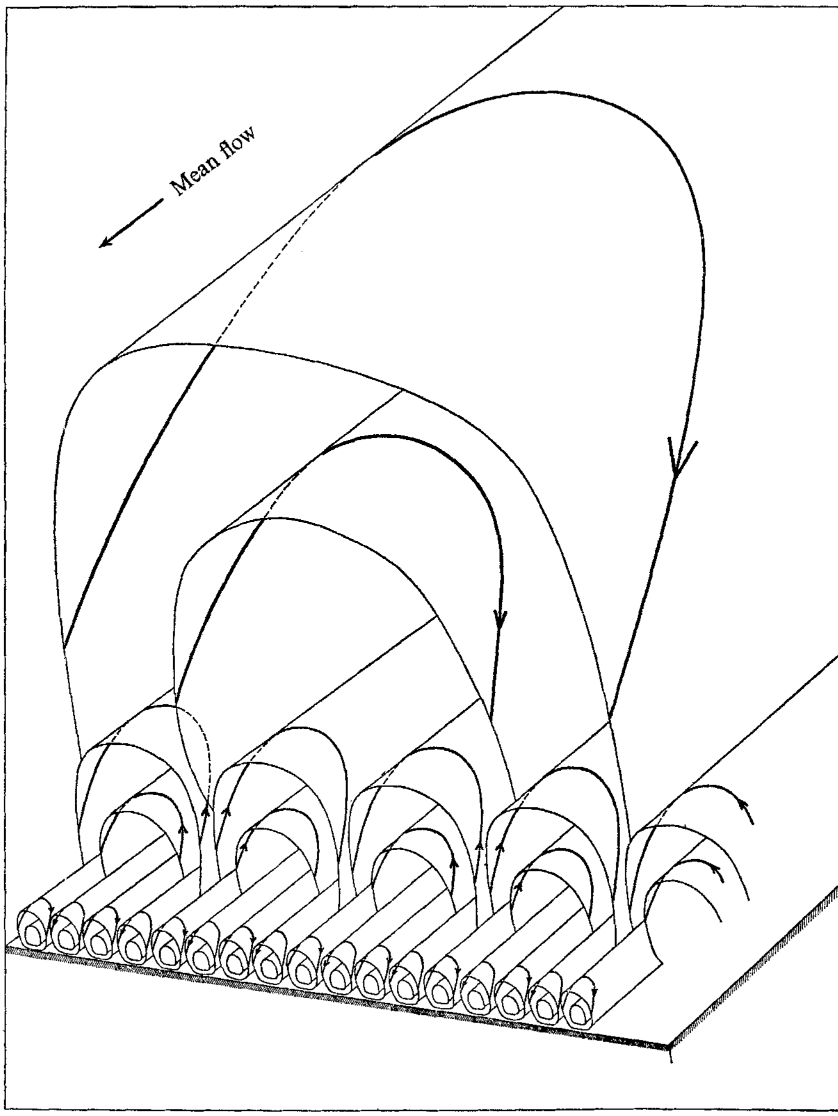

FIGURE 3. Qualitative sketch of the boundary-layer region of the vector field yielding maximum transport of momentum.

structure of the minimizing vector field has been sketched in figure 3. The thicknesses of subsequent boundary layers differ always by a factor of about 4, which becomes the exact factor in the asymptotic case of μ tending to infinity. In this case, the minimum among the functions $B^{(N)}(\mu)$ is determined by

$$4^N = (h_0\mu)^{\frac{1}{2}}(\sigma/\beta)^{\frac{3}{4}}\, h_1/4^{\frac{4}{3}}\beta. \tag{33}$$

Accordingly the following expressions are obtained for the thicknesses d_n and the wave-numbers α_n of the boundary layers in the limit $\mu \to \infty$:

$$\left.\begin{aligned} d_n &\equiv \mu^{-r_n}(b_n b_{n+1}^2)^{-\frac{1}{3}} \to h_1/\beta 4^{n+1}, \quad \text{for} \quad n = 1, \ldots, N-1, \\ d_N &\equiv \mu^{-r_N}(b_N h_0)^{-\frac{1}{3}} \to h_1(\sigma/\beta)^{\frac{1}{2}}/\beta 4^{N+1}, \\ \alpha_n &= \mu^{q_n/2} b_n \to 4^{n+\frac{1}{3}}\beta/h_1, \quad \text{for} \quad n = 1, \ldots, N. \end{aligned}\right\} \tag{34}$$

The discussion of the structure of the extremalizing field will be resumed in § 6, where turbulent pipe flow is considered, in which case experimental observations offer additional possibilities for a comparison.

5. Turbulent channel flow

In the case when the two plates considered in § 2 are at rest, and the flow is driven by a constant pressure gradient $-A_p \mathbf{i}$, the relation that can be derived in place of relation (2) is

$$\langle |\nabla \times \hat{\mathbf{v}}|^2 \rangle + \langle (\overline{\hat{\mathbf{u}}\hat{w}})^2 \rangle - \langle \hat{\mathbf{u}}\hat{w} \rangle^2 - A_p \langle \hat{\mathbf{u}} \cdot \mathbf{i}\hat{w}z \rangle = 0. \tag{35}$$

The symmetry of the problem suggests that $\langle \hat{\mathbf{u}}\hat{w} \rangle$ is vanishing. Otherwise, the equation for the mean flow

$$\frac{d}{dz}\mathbf{U} = \overline{\hat{\mathbf{u}}\hat{w}} - \langle \hat{\mathbf{u}}\hat{w} \rangle - A_p z \mathbf{i} \tag{36}$$

would lead to an asymmetric profile. For simplicity we shall neglect the term $\langle \hat{\mathbf{u}}\hat{w} \rangle$ in the following discussion. When $\langle \hat{\mathbf{u}}\hat{w} \rangle$ is retained the analysis will yield the same results, with $\langle \mathbf{u}w \rangle = 0$ as a property of the extremalizing solutions.

For the Reynolds number *Re*, which is defined as the flow rate $\langle \mathbf{i} \cdot \mathbf{U} \rangle$ in the direction opposite to the pressure gradient, the relation

$$Re \equiv \langle \mathbf{U} \rangle \cdot \mathbf{i} = -\left\langle z \frac{d}{dz}\mathbf{U} \right\rangle \cdot \mathbf{i} = \tfrac{1}{12}A_p - \langle \hat{\mathbf{u}}\hat{w}z \rangle \cdot \mathbf{i} \tag{37}$$

holds. In the previous analysis of channel flow in (I), the minimum of a functional corresponding to the pressure gradient A_p was considered as a function of the parameter $\mu = \langle \mathbf{u} \cdot \mathbf{i}wz \rangle$. The present variational problem differs in that μ corresponds to $\langle u\mathbf{i}w\sqrt{12}z \rangle$, and in that μ has been subtracted in the definition of the functional:

Given $\mu \geqslant 0$ find the minimum $R(\mu)$ of the functional

$$\mathscr{R}(\mathbf{v}, \mu) \equiv \frac{\langle |\nabla \times \mathbf{v}|^2 \rangle}{\langle \mathbf{u} \cdot \mathbf{i}\sqrt{12}zw \rangle} + \mu \frac{\langle (\overline{\mathbf{u}w})^2 \rangle - \langle \mathbf{u} \cdot \mathbf{i}w\sqrt{12}z \rangle^2}{\langle \mathbf{u} \cdot \mathbf{i}w\sqrt{12}z \rangle^2} \tag{38}$$

among all vector fields $\mathbf{v}$ that satisfy conditions (3), (4).

$R(\mu)$ provides a lower bound for $\sqrt{12} \cdot Re$ at a given value μ of $\langle \mathbf{i} \cdot \hat{\mathbf{u}}\hat{w}\sqrt{12}z \rangle$, according to relations (35) and (37). This lower bound holds also at a given value A_p of $P(\mu) \equiv \sqrt{12}(R(\mu) + \mu)$, since it can be shown in the usual way that $R(\mu)$, like $P(\mu)$, is a positive monotonically increasing function of μ, with the property that the inverse function exists.

The definition (38) of the variational functional has the advantage that the analysis of §3 can be applied directly. As in the Couette case, we introduce the hypothesis that the minimum of (38) is reached for solutions $\mathbf{v}$ with vanishing x-dependence. Using the representation (8), (9) and the relation (12), and assuming $\overline{u_y w} = 0$ as in the Couette case, we can rewrite $\mathscr{R}(\mathbf{v};\mu)$ in the form

$$\mathscr{R}(v,\theta;\mu) \equiv 2\frac{\langle|\mathbf{k}\times\nabla\nabla^2 v|^2\rangle^{\frac{1}{2}}\cdot\langle|\nabla\theta|^2\rangle^{\frac{1}{2}}}{\langle w\theta\sqrt{12}\,z\rangle}+\mu\frac{\langle(\overline{w\theta}-\sqrt{12}\,z\langle w\theta\sqrt{12}\,z\rangle)^2\rangle}{\langle w\theta\sqrt{12}\,z\rangle^2},$$

which is identical with the definition (13) in the special case $h(z) = \sqrt{12}\,z$. The fact that we have assumed $h(\frac{1}{2}) = h(-\frac{1}{2})$ in §3 does not cause any difficulty, since the analysis applies to the present case when θ is replaced by $-\theta$ for $z < 0$. A minor problem arises from the fact that the solution (25*c*) for $\tilde{w}_1$, $\tilde{\theta}_1$,

$$\tilde{w}_1 = |\sqrt{12}\,z|^{\frac{1}{2}}, \quad \tilde{\theta}_1 = \pm|\sqrt{12}\,z|^{\frac{1}{2}}, \tag{39}$$

leads to a divergent contribution in the dissipation term. This divergence has an artificial character, because the z-dependence of $\tilde{w}_1$ and $\tilde{\theta}_1$ in the interior has been neglected in the boundary-layer approximation (19), from which the solution (39) results. By introducing a fourth-order polynomial for $\tilde{w}_1$ in the neighbourhood of $z = 0$, and assuming $\tilde{\theta}_1 = \sqrt{12}\,z/\tilde{w}_1$, it can be demonstrated that the additional contribution in the dissipation term is indeed small compared to $b_1^2\mu^{q_1}$. In this connexion, it can also be shown that the fact that $\tilde{w}_1$ enters the dissipation term with higher derivatives than $\tilde{\theta}_1$ causes a preference of $\tilde{\theta}_1$ rather than $\tilde{w}_1$ as antisymmetric function in z, which justifies the above choice. Accordingly, the class $\hat{R}^{(N)}(\mu)$ of the minima of $\hat{\mathscr{R}}$ for solutions of the form (16) is given by the expression (30), with $h_0 = \sqrt{3}$, $h_1 = \frac{1}{2}$. We close the discussion of turbulent channel flow at this point and turn to the closely related case of pipe flow, for which more extensive experimental observations are available for the comparison with the solution of the variational problem.

6. Turbulent pipe flow

For the discussion of turbulent flow in a pipe we introduce a cylindrical system of co-ordinates (r,ϕ,x), and assume that the inner surface of the pipe is given by $r = 1$. The equations corresponding to (35) and (36) in the case of channel flow are

$$\left.\begin{aligned}\langle|\nabla\times\hat{\mathbf{v}}|^2\rangle+\langle(\overline{\hat{v}_r\hat{v}_\phi})^2+(\overline{\hat{v}_r\hat{v}_x})^2\rangle-\tfrac{1}{2}A_p\langle\hat{v}_r\hat{v}_x r\rangle = 0,\\ \frac{d}{dr}U_\phi = \frac{1}{r}\overline{\hat{v}_r\hat{v}_\phi}, \quad \frac{d}{dr}U_x = \overline{\hat{v}_r\hat{v}_x}-\tfrac{1}{2}A_p r.\end{aligned}\right\} \tag{40}$$

The average indicated by a bar is taken over surfaces r = const., the angular brackets indicate the average over the infinite cylindrical volume contained by the pipe. A_p is the absolute value of the pressure gradient in the direction opposite to the x co-ordinate. The Reynolds number is defined by

$$Re \equiv \langle U_x\rangle = -2\int_0^1 \frac{r^2}{2}\frac{d}{dr}U_x dr = \tfrac{1}{8}A_p - \tfrac{1}{2}\langle\hat{v}_r\hat{v}_x r\rangle.$$

The following variational problem provides a lower bound for Re at a given value μ of $\langle \hat{v}_r \hat{v}_x r \rangle$.

Determine the minimum $P(\mu)$ of the functional

$$\mathscr{P}(\mathbf{v};\mu) \equiv \frac{\langle |\nabla \times \mathbf{v}|^2 \rangle}{\langle v_r v_x \sqrt{2r} \rangle} + \mu \frac{\langle (\overline{v_r v_x})^2 + (\overline{v_r v_\phi})^2 \rangle - \langle v_r v_x \sqrt{2r} \rangle^2}{\langle v_r v_x \sqrt{2r} \rangle^2} \tag{41}$$

at a fixed value of μ among all vector fields $\mathbf{v}$ that vanish at $r = 1$ and satisfy the equation of continuity $\nabla \cdot \mathbf{v} = 0$. The factor $\sqrt{2}$ has been introduced to give $\langle (\sqrt{2r})^2 \rangle = 1$. $P(\mu)$ provides the lower bound for $\sqrt{8}\,Re$ at a given value of $\langle r\hat{v}_r \hat{v}_x \rangle$, or at given value of A_p, as in the case of channel flow. For the solution of the variational problem, the hypothesis is used again that the absolute minimum of the functional (41) is reached by vector fields $\mathbf{v}$ independent of x. In the present case, however, the hypothesis is not correct for all values of μ, as has been shown by Joseph & Carmi (1969), who considered the case $\mu = 0$. Yet in the same paper it was also shown that the exact solution depends only slightly on x. Hence, we expect that the x-independent solution will approximate the absolute minimum of the functional (41) closely, if not exactly. We use the hypothesis to eliminate the equation of continuity by the introduction of the new variables v, θ:

$$v_r = -\frac{\partial^2}{\partial \phi^2} v \equiv w, \quad v_\phi = \frac{\partial^2}{\partial \phi\, \partial r} rv, \quad v_x = \theta.$$

As in the preceding cases, it is anticipated that $\overline{v_r v_\phi}$ vanishes. Using the following necessary condition for a minimum of the dissipation term,

$$\begin{aligned}\langle |\nabla \times \mathbf{v}|^2 \rangle &= \langle (Lrv)^2 \rangle + \left\langle \left(\frac{1}{r}\frac{\partial}{\partial \phi}\theta\right)^2 + \left(\frac{\partial}{\partial r}\theta\right)^2 \right\rangle \\ &= 2\langle (Lrv)^2 \rangle^{\frac{1}{2}} \left\langle \left(\frac{1}{r}\frac{\partial}{\partial \phi}\theta\right)^2 + \left(\frac{\partial}{\partial r}\theta\right)^2 \right\rangle^{\frac{1}{2}},\end{aligned}$$

we consider in place of (41) the functional,

$$\mathscr{P}(v,\theta;\mu) = 2\,\frac{\langle (Lrv)^2 \rangle^{\frac{1}{2}} \left\langle \left(\frac{1}{r}\frac{\partial}{\partial \phi}\theta\right)^2 + \left(\frac{\partial}{\partial r}\theta\right)^2 \right\rangle^{\frac{1}{2}}}{\langle w\theta \sqrt{2r} \rangle} + \mu \frac{\langle (\overline{w\theta} - \sqrt{2r}\langle \overline{w\theta}\sqrt{2r} \rangle)^2 \rangle}{\langle w\theta \sqrt{2r} \rangle^2}, \tag{42}$$

which is homogeneous of degree zero with respect to v as well as to θ. L is used as abbreviation for

$$\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right)\frac{\partial}{\partial \phi}.$$

The solution of the variational problem will be assumed in the form (16), with the difference that r replaces z, and that ϕ_n is a function of ϕ in the present case, satisfying the relation

$$\frac{d^2}{d\phi^2}\phi_n = -\alpha_n^2 \phi_n.$$

Apart from the fact that a different geometry is considered, the functional (42) is identical with the functional (13) if $h(z)$ is identified with $\sqrt{2r}$. The effects of the geometry become less important when the boundary-layer approximation, with

$$\zeta_n \equiv (1-r)\mu^{r_n} \quad \text{for} \quad n = 1, \ldots, N$$

as boundary-layer co-ordinates, is considered. The boundary-layer approximation $\hat{\mathscr{P}}$ of the functional (42) is described by the expression (19) if the interior parts of the dissipation terms,

$$\mu^{q_1} b_1^2 \langle \tilde{w}_1^2 \rangle, \quad \mu^{q_1} b_1^2 \langle \tilde{\theta}_1^2 \rangle,$$

are replaced by

$$\mu^{q_1} b_1^2 \langle \tilde{w}_1^2 r^{-2} \rangle, \quad \mu^{q_1} b_1^2 \langle \tilde{\theta}_1^2 r^{-2} \rangle,$$

respectively.

Thus the analysis of § 3 can be applied directly in the present case by specifying

$$h_0 = \sqrt{2}, \quad h_1 = 2. \tag{43}$$

In doing so, we have regarded the wave-number α_n as a continuous parameter, although only integer values are admissible. More detailed calculations show that the functions $\hat{P}^{(N)}(\mu)$, given by the expression (30) with the specification (43), change by a negligible amount if the analysis is limited to integer values α_n. This fact holds, even though the interior wave-number α_1 of the extremalizing solution is only two, as the relations (34), (43) suggest.

The experimental observations are often plotted in terms of the friction coefficient $\lambda = 4 \cdot A_p \cdot Re^{-2}$. At a given Reynolds number, the inverse function $\mu(P)$ of $P(\mu)$ provides an upper bound for λ,

$$\lambda \leqslant \frac{32}{Re} + \frac{8\sqrt{2}\,\mu(\sqrt{8}\,Re)}{Re^2}. \tag{44}$$

The upper bound is given throughout the region of interest by the functions $\hat{\mu}^{(N)}(P)$ with rather high values of N for which the asymptotic formula (31) holds. Accordingly, the upper bound for λ is essentially constant, like the upper bound derived without the constraint of the continuity equation (Busse 1968). For a comparison, experimental data by Nikuradse (1932) are shown in figure 4. Since the bounds have been derived under the assumption of large μ, the first term on the right side of (44) has been neglected in the drawing.

Extensive data about the structure of turbulent pipe flow are available and invite a comparison with the structure of the minimizing solution of the variational problem. Because of the incomplete representation of this solution by the boundary-layer theory, it is difficult to obtain more than a qualitative comparison. Asymptotically the minimizing solution has the property that it depends only on $\mu^{\frac{1}{2}}$ near the wall. This behaviour is reflected in the wall-proximity law of Prandtl, which states that the turbulent flow becomes independent of the Reynolds number if the friction velocity U_τ and the length U_τ/ν are used as scales for the velocity and the distance from the wall. In the dimensionless description used in this paper, the friction velocity $\hat{U}_\tau$ is defined by

$$\hat{U}_\tau = \left(-\frac{dU_x}{dr}\bigg|_{r=1} \right)^{\frac{1}{2}} = (\tfrac{1}{2}A_p)^{\frac{1}{2}} \approx 2^{\frac{1}{4}} \langle \hat{v}_r \hat{v}_x \sqrt{2}\, r \rangle^{\frac{1}{2}}, \tag{45}$$

where the last equality is approached in the case of fully developed turbulence when the transport of momentum is carried almost entirely by the turbulent motions. In figure 5 the asymptotic boundary-layer dependence of θ, given by

$$\theta_n(z) = \mu^{4^{-n-1}}\sqrt{h_0}\left(\frac{b_{n+1}}{b_n}\right)^{\frac{1}{3}}\hat{\Theta}(d_n^{-1}z) \to 4^{\frac{1}{3}}\hat{\Theta}(zd_n^{-1})\sqrt{h_0}, \quad \text{for} \quad n = 1, \ldots, N-1,$$

$$\theta_N(z) = \mu^{4^{-N-1}}(h_0^{\frac{3}{2}}/b_N)^{\frac{1}{3}}\Theta(zd_N^{-1}) \to (4^{\frac{4}{3}}h_0^{\frac{3}{2}}\beta/\sigma)^{\frac{1}{4}}\Theta(zd_N^{-1})$$

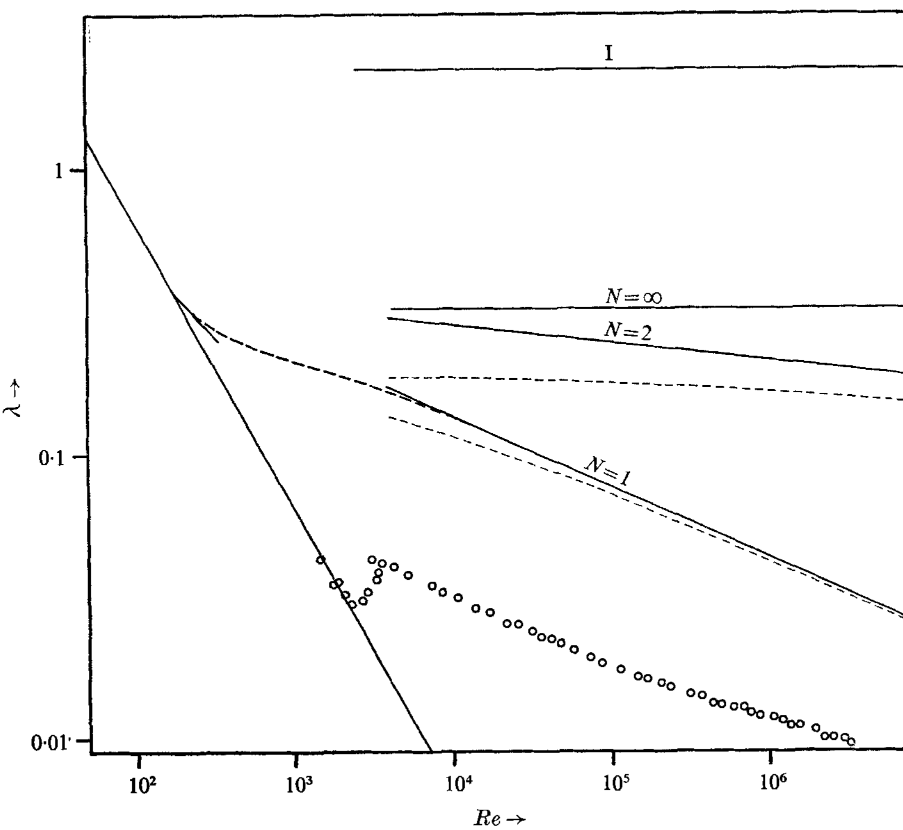

FIGURE 4. The upper bound for the friction coefficient λ of turbulent pipe flow in comparison with experimental data by Nikuradse (1932). The line labelled I denotes the asymptotic bound derived without the constraint of the continuity equation.

has been plotted together with the observed similarity dependence for the streamwise component of the fluctuating velocity field. The functions $\hat{\Theta}$ and Θ have been taken from (C) and from Howard's paper. The velocity component normal to the wall shows a similar correspondence. Because of the lack of correlation, the r.m.s. values of the turbulent velocity field normalized by U_τ have much higher amplitudes than w and θ in the corresponding normalization by $\hat{U}_\tau = (\sqrt{2}\mu)^{\frac{1}{4}}$.

In describing the structure of turbulent pipe flow, Laufer (1955) arrives at the following conclusions:

(i) Throughout the whole cross-section, with the exception of the centre region, the rate of energy production at a point is approximately balanced by the rate of energy dissipation.

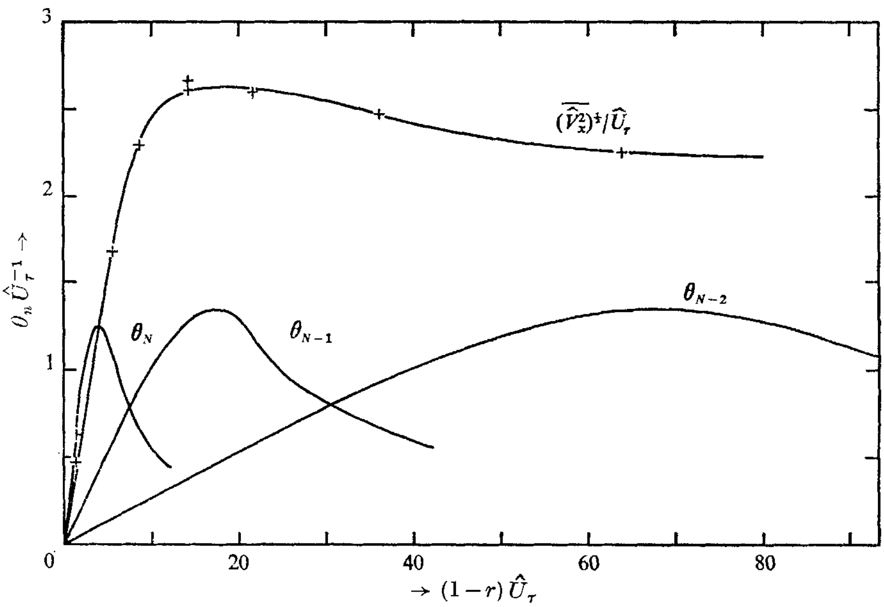

FIGURE 5. The boundary layer dependence of extremalizing field $\theta = \sum_n \theta_n$ in comparison with the r.m.s. value of the fluctuations of the streamwise velocity component $\hat{v}_x/\hat{U}_\tau$ measured by Laufer (1955).

(ii) All the various energy rates reach a sharp maximum near the edge of the laminar sublayer.

These statements hold equally for the minimizing solution of the variational problem. The first statement corresponds to the balance described by the Euler equations, the second statement reflects the dominating contribution in the functional from the Nth boundary layer. A closer inspection shows that the dissipation rate of the fluctuating velocity field is inversely proportional to the distance from the wall, like the contributions of the boundary layers to the dissipation term in the functional (19). The observed fact that the total dissipation of the fluctuating velocity field is approximately equal to the dissipation of the mean flow corresponds to the basic balance between the two terms of the functional (42).

The profile of the mean flow does not show a similarity between the realized turbulence and the minimizing solution as clearly as in the Couette case. The exact dependence on r will be strongly influenced by the higher-order corrections, which have been neglected in the boundary-layer approximation. We mention in particular the removal of the divergence of $(d/dr)\tilde{\theta}_1$, $(d^2/dr^2)\tilde{w}_1$ at the centre, which was discussed in the analogous case of channel flow. According to (24b), (25*c*) and (40), a parabolic profile for the minimizing solution is obtained asymptotically:

$$\frac{dU_x^{(N)}}{dr} = \mu(\overline{w^{(N)}\theta^{(N)}} - \sqrt{2r}\langle w^{(N)}\theta^{(N)}\sqrt{2}\,r\rangle) - \sqrt{2}\,r\hat{P}^{(N)}(\mu)$$
$$\approx -\tfrac{1}{4}\sqrt{2}\,r\hat{P}^{(\infty)}(\mu) = -r\langle U_x^{(\infty)}\rangle.$$

This profile joins the boundary close to the wall at $\frac{3}{5}$ of its maximal value at the pipe centre, which is about the same region as in the case of the realized turbulence. In the case of channel flow, the value $\frac{3}{5}$ is replaced by $\frac{2}{3}$.

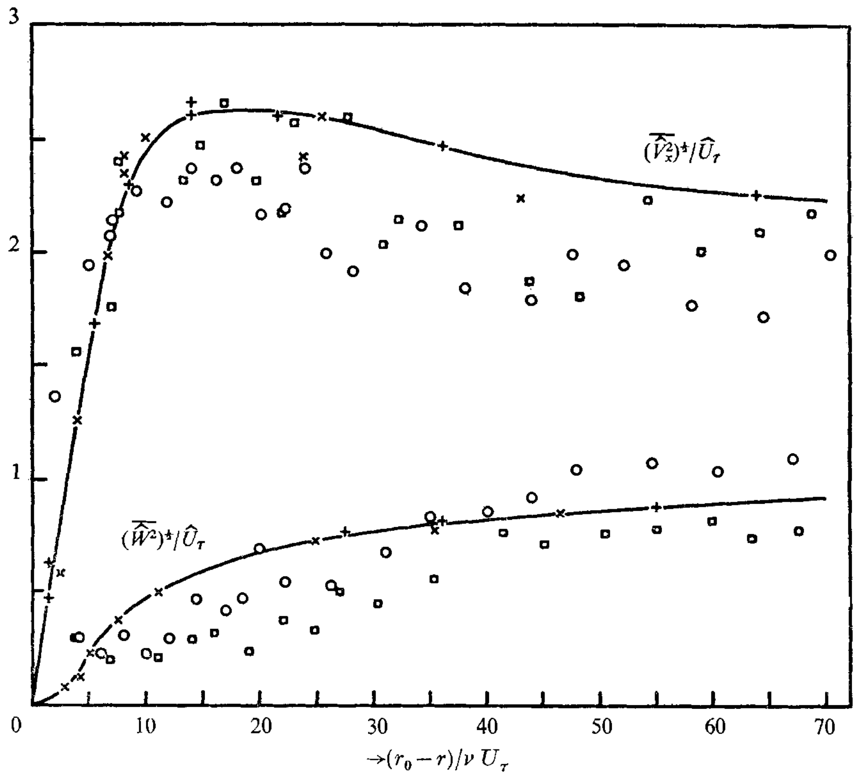

FIGURE 6. R.m.s. values of the fluctuating component of the velocity in streamwise direction, $\hat{v}_x/\hat{U}_\tau$, and normal to the wall, $\hat{w}/\hat{U}_\tau$, measured by Laufer (1955) at $Re = 2{\cdot}5\times10^4(+)$ and $Re = 2{\cdot}5\times10^5(\times)$. For comparison the r.m.s. values of the temperature fluctuations $\hat{\theta}$ and of the vertical velocity component $\hat{w}$, which were measured in turbulent thermal convection by Deardorff & Willis at $Re = 2{\cdot}5\times10^6(\bigcirc)$ and $Re = 1{\cdot}0\times10^7(\square)$, are plotted in units resulting from the correspondence of the variational problems.

We have pointed out at several instances the similarity between the variational problem in the case of thermal convection and in the case of shear flow. If in fact the physically realized turbulence approaches the structure of optimal transport, a correspondence between the data of turbulent convection and of turbulent shear flow should exist. To exhibit this correspondence, measurements by Deardorff & Willis (1967) of the r.m.s. values of the fluctuating temperature field $\hat{\theta}$

and of the vertical velocity field $\hat{w}$ in turbulent convection have been plotted in figure 6 on top of a figure taken from Laufer's paper. No arbitrary adjustment of the scales has been assumed.

In the present usage, the parameter μ corresponds to $\hat{U}_\tau^2/\sqrt{2}$ according to relation (45), while the analogous parameter $\mu^{(c)}$ in the variational problem of (C) corresponds to the dimensionless turbulent heat transport H_c. Accordingly, the characteristic boundary-layer scale $d_N = (4^{\frac{4}{3}}\beta/\sigma)^{\frac{1}{4}}/U_\tau$ defined by the asymptotic expressions (33), (34) has been identified with the corresponding scale $d_N^{(c)} = (\beta 4^{\frac{7}{3}}/H_c)^{\frac{1}{3}}$ given by the asymptotic extremalizing solution in the case of thermal convection. Similarly, the scales for the amplitudes of $\hat{\theta}$ and $\hat{w}$ have been determined. Although the experimental scatter of the data in thermal convection is rather high, it can be concluded from figure 6, first, that a similarity law for turbulent convection is valid in analogy to Prandtl's law of the wall in turbulent shear flow, and secondly, that both similarity laws are essentially identical, if the units suggested by the extremalizing transport mechanism are used.

7. The limiting stability property of turbulent shear flow

The similarity, which was found in the comparison between the observed structure of turbulence and the minimizing solution of the variational problem, suggests that the realized turbulent shear flow represents the flow with maximum momentum transport, or with maximum dissipation at a given Reynolds number among all possible solutions of the Navier–Stokes equations of motion. We do not think that this fact indicates the existence of an extremum principle for turbulent flow which is valid in an exact sense. The tendency towards the property of maximum momentum transport seems rather to be the consequence of the instability of the laminar sublayer, as the following consideration suggests:

A laminar flow usually becomes unstable if the characteristic Reynolds number exceeds a certain critical value R_c. If a criterion of this kind is applied to the laminar sublayer adjacent to the wall in which the momentum transport M is carried by viscous stresses, we obtain

$$Re\,\delta \gtrsim R_c$$

as the criterion for instability. δ denotes the thickness of the sublayer, Re can be used as an estimate of the velocity change across the sublayer. Since δ is related to the momentum transport, $M \approx Re/\delta$, the criterion for instability can be rewritten in the form,

$$Re^2 \gtrsim R_c M.$$

Accordingly, the laminar sublayer will be unstable unless the momentum transport grows like Re^2. On the other hand, we have found that M can not grow stronger than proportional to Re^2 asymptotically. Thus, the momentum transport is forced to grow like Re^2, which appears to be in agreement with the experimental observation in the Couette case. The measurements of turbulent pipe flow seem to indicate a logarithmic decrease of the friction coefficient in place of a constant asymptotic value. Since the velocity of the mean flow just outside the laminar sublayer is rather small compared to Re, the above instability mechanism may still be relevant without forcing the momentum transport to approach the dependence of the upper bound.

The idea that turbulent flow can be characterized by the limitations of an optimal transport mechanism on one hand, and by a stability criterion on the other, is related to earlier theories of turbulence. We mention in particular Malkus' (1956) theory, which was successful in deriving a number of features of turbulent shear flow from the hypothesis that turbulence can be characterized by a maximum dissipation at a given Reynolds number. In place of a relation of the form (2) Malkus has used the following two additional hypotheses as constraints: first, that the mean flow is stable in terms of the linear stability analysis described by the Orr–Sommerfeld equation, and secondly, that the smallest scale of motion corresponds to marginally stable disturbances of the mean flow. The latter hypothesis is related to the above interpretation of turbulent shear flow in terms of the stability of the viscous sublayer. The first hypothesis, however, is not reflected in the structure of the extremalizing field which exhibits profiles of the mean flow of the same form as the profile of the unstable laminar flow.

The extremalizing field of the variational problem shows, according to (34), an obvious relation to Prandtl's mixing-length theory, which states that the characteristic scale of turbulent eddies increases proportional to the distance from the boundary. An essential difference, however, is the fact that the structure with the property of optimal transport has discrete scales. These discrete scales may be present in turbulent shear flow, although it will be difficult to demonstrate their existence experimentally. In the case of turbulent thermal convection, experimental evidence for discrete scales has been found by Deardorff & Willis (1967).

The experiments on thermal convection exhibit yet another related phenomenon, namely, discrete transitions in the dependence of the heat transport on the Rayleigh number. In the regime of turbulent convection, the transitions were discovered by Malkus (1954). The transitions parallel qualitatively the transitions by which the solutions of the Euler equations $v^{(N)}$, $\theta^{(N)}$ take turns in providing the upper bound for the heat transport. The close relation between the turbulent processes of heat and momentum transport suggests that the phenomenon of discrete transitions corresponding to distinct instabilities of the laminar sublayer exists in turbulent shear flow, too.

8. Conclusion

The bounds that have been derived in the preceding sections depend strongly on the region adjacent to the rigid boundary. Neglecting the effects of the geometry of the interior, we may conclude that the exchange of momentum between turbulent shear flow and a rigid boundary grows asymptotically proportional to the square of the Reynolds number or less. This conclusion implies the negative result that the upper bound on the momentum transport obtained from the variational problem, without the constraint of the continuity equation or by even simpler estimates, cannot easily be improved upon with respect to its qualitative dependence. Quantitatively, the continuity equation leads to an improvement by a factor of about 7 of the result without this equation. The most interesting consequence of the equation of continuity as constraint in the variational problem is the fact that the extremalizing field exhibits a cascade of discrete scales depending on the distance from the boundary layer. The similarity between the variational problems for turbulent shear flow and for turbulent thermal convection suggests that the multiple boundary-layer structure is a common feature of the vector fields extremalizing the turbulent transport of a quantity from a rigid boundary. The comparison with the experimental observations indicates in all cases considered so far that the realized turbulent flow tends to approach the structure of the extremalizing transport mechanism. The extremalizing vector field will show an even closer relation to the observed turbulence when additional constraints derived from the basic equations are introduced in the variational problem. Finally, the realized turbulent flow may be approached in this process, although the Euler equations are likely to become almost as difficult to solve as the original equations of motion.

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